Bargaining Problems with Arbitrary Reference Points

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Abstract

A number of authors have enriched Nash's original bargaining problem through the introduction of an additional point, which we call a salient point. When the salient point is restricted to lie in the bargaining set, it is interpreted as a reference point, the outcome, perhaps, of a bargain from an earlier period. When restricted to lie outside the bargaining set, it is interpreted as a claim on the resources being shared. Here we remove the restriction that the salient point must lie either in or outside the bargaining set to obtain a new model we call the *Bargaining Problem with a Salient Point*. We then generalize the Tempered Aspirations solution (Balakrishnan, Gómez, and Vohra 2011) and study its properties in this new setting.

Key words: axiomatic bargaining; reference point; claims point; Tempered Aspirations solution

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1 Introduction

In Nash's (1950) traditional bargaining framework, the outcome of a negotiation is a function only of the bargaining set, S, and the disagreement point $d \in S$. The first is the set of achievable utility profiles whilst the second represents the utilities the parties receive in the event a bargain is not struck. It has been noted by a number of writers that other aspects of the environment influence the outcome of the bargain. For example, the precedent set by a similar bargain from an earlier period. The practice of pattern bargaining employed by some trades unions is another example. The agreement struck with one employer in the industry becomes the reference point for agreements with other employers. To accommodate this some authors have enriched Nash's framework through the introduction of an additional point, r, we call a salient point. We call the resulting model the *Bargaining Problem with a Salient Point*.

In Gupta and Livne (1989), r is restricted to lie in S. They interpret r to be an intermediate agreement which facilitates conflict resolution, such as last period's outcome. They also propose and characterize a solution to the bargaining problem that depends on S, d and r. However, it is not clear why an outcome reached in earlier period need be feasible at some later time. Perhaps, last periods outcome is infeasible because of straitened circumstances. Alternatively, r could represent agreements made before the set of feasible outcomes was realized. Thus, one can imagine situations where $r \notin S$. Chun and Thomson (1992) consider just this case and propose and characterize a solution to the bargaining problem. The solution in Gupta and Livne (1989) differs from the one proposed in Chun and Thomson

(1992). We argue that there is no reason why a solution to the bargaining problem that is compelling when $r \in S$ should lose its appeal when $r \notin S$. In this paper we impose no a priori restriction on the location of r.

After describing the new model and solution concept in Section 2, in Section 3 we show how previous works fit within our framework. In Section 4, we conclude by classifying different solution concepts in the literature according to the main features of our model.

2 The Model

2.1 Preliminaries

Let $n > 1$ be a fixed natural number and define $N = \{1, \ldots, n\}$. For any $x \in \mathbb{R}^n$, $i \in N$ and $t \in \mathbb{R}$, let (t, x_{-i}) represent the vector $y \in \mathbb{R}^n$ such that $y_i = t$ and $y_j = x_j$ for any $j \neq i$. Also, if $t > 0$, let $\frac{x}{t}$ abbreviate $\frac{1}{t}x$. Vector inequalities are treated as follows. Given $x, y \in \mathbb{R}^n$ we write $x \geq y$ if $x_i \geq y_i \ \ \forall i \in N$, $x > y$ if $x \geq y$ but $x \neq y$, and $x \gg y$ if $x_i > y_i \ \ \forall i \in N$. Let Y be any subset of $\mathbb{R}^n$. Y is said to be *convex* if for all $y_1, y_2 \in Y$ and all $\lambda \in [0, 1]$, $\lambda y_1 + (1 - \lambda) y_2 \in Y$. Its *convex hull*, denoted by $con(Y)$, is defined as the intersection of all convex sets containing Y. Y is called *comprehensive* if for every $x \in \mathbb{R}^n$ the fact that there is a $y \in Y$ such that $y \geq x$ implies $x \in Y$. The *convex and comprehensive hull* of Y, $cch(Y)$, is the intersection of all convex and comprehensive sets containing Y.

2.2 Bargaining Problems

An *n-person bargaining set* is any non-empty, convex, comprehensive and closed set $S \subseteq \mathbb{R}^n$ that is *bounded above* in the sense that there is a $p \in \mathbb{R}^n_{++}$ and a $w \in \mathbb{R}$ such that $\sum_{i \in N} p_i x_i \leq w$ for all $x \in S$. The bargaining set represents all the utility vectors that can be achieved by the players bargaining among themselves.

Convexity presumes that randomization over the outcomes is possible. The comprehensiveness condition reflects the possibility of free disposal of utility. For any n-person bargaining set S, define its *Pareto-optimal set* as $PO(S) = \{y \in S \mid x > y \text{ implies } x \notin S\}$. Similarly, its *weakly Pareto-optimal set* is defined as $WPO(S) = \{y \in S \mid x \gg y \text{ implies } x \notin S\}$.

An *n-person Bargaining Problem* (Nash 1950) is a pair (S, d) such that S is an n-person bargaining set, $d \in S$, and there exists an $x \in S$ satisfying $x \gg d$. The point d is called the *disagreement point* and represents the utility obtained by the bargainers if no agreement is reached. The *ideal point* of the problem (S, d) represents bargainers' expectations before coming to the negotiation table and is defined by $z_i(S, d) = \max\{t \in \mathbb{R} \mid (t, d_{-i}) \in S\}$ for any $i \in N$. Denote the family of all n-person bargaining problems by Σ_0^n.

A *solution concept for n-person Bargaining Problems* or simply a *solution on* Σ_0^n is a function $\psi : \Sigma_0^n \longrightarrow \mathbb{R}^n$ that associates with each $(S, d) \in \Sigma_0^n$ a unique outcome $\psi(S, d) \in S$. For example, the *Kalai-Smorodinsky solution* (Kalai and Smorodinsky 1975) is defined for every $(S, d) \in \Sigma_0^n$ as $KS(S, d) = (1 - \lambda^*)d + \lambda^* z(S, d)$ where $\lambda^* = \max\{\lambda \in [0, 1] \mid (1 - \lambda)d + \lambda z(S, d) \in S\}$.

2.3 Bargaining Problems with a Salient Point

Definition 2.1 *We define an* **n-person Bargaining Problem with a Salient Point** *as a triple* (S, d, r) *where* (S, d) *is an* n*-person Bargaining Problem and the salient point* $r \in \mathbb{R}^n$ *satisfies* $r \geq d$.

If $r \in S$ we obtain a *Bargaining Problem with a Reference Point* (Gupta and Livne 1988). If $r \notin S \setminus PO(S)$ and for every $i \in N$ there is an $x \in S$ such that $x_i \geq r_i$, we can interpret our setting as a *Bargaining Problem with Claims* (Chun and Thomson 1992). In the latter case r plays the role of the claims vector.

Denote the family of all n-person bargaining problems with a salient point by Σ^n. A *solution concept for an* n*-person Bargaining Problem with a Salient Point* or simply a *solution on* Σ^n is a function $\phi : \Sigma^n \longrightarrow \mathbb{R}^n$ that associates with each triple $(S, d, r) \in \Sigma^n$ a unique outcome $\phi(S, d, r) \in S$.

3 Generalizing the Tempered Aspirations Solution

Definition 3.1 *Define for every* $(S, d, r) \in \Sigma^n$ *the* **aspiration** *of bargainer* $i \in N$ *as*

$$a_i(S, d, r) = \max\{r_i, \max\{t \in \mathbb{R} \mid (t, r_{-i}) \in S\}\}$$

and let $a(S, d, r)$ *be the* **aspiration vector** *of problem* $(S, d, r) \in \Sigma^n$.

The aspiration vector is a generalization of Chun and Thomson's (1992) claims/expectations vector to our more general setting. Bargainer i aspires to obtain her claim, r_i, plus any surplus that remains after satisfying all other bargainers' claims. If the salient point r is outside the bargaining set, claims (expectations) and aspirations are identical. If not, the aspiration vector captures the idea of bargainers increasing their aspirations when they know the size of the pie is larger.

Definition 3.2 *The* **Generalized Tempered Aspirations (GTA) solution** *is defined as*

$$GTA(S,d,r) = \lambda^* a(S,d,r) + (1-\lambda^*)d$$

where $\lambda^* = \max\{\lambda \in [0,1] \mid \lambda a(S,d,r) + (1-\lambda)d \in S\}$ *(see Figure* 1*).*

If a bargaining problem is translated so that the disagreement point is at the origin, our proposed solution is the only point along the frontier of S proportional to the aspiration vector.

4 Characterization

4.1 The Axioms

Weak Pareto Optimality (WPO): A solution ϕ on Σ^n satisfies WPO if $\phi(S,d,r) \in WPO(S)$ for every $(S,d,r) \in \Sigma^n$.

Let $\Pi(N)$ be the set of permutations of the set N. For any $\pi \in \Pi(N)$ and any $x \in \mathbb{R}^n$ define $\pi(x) \in \mathbb{R}^n$ as the vector such that for every $i \in N$, $(\pi(x))_{\pi(i)} = x_i$. For any $X \subseteq \mathbb{R}^n$ define $\pi(X) = \{\pi(x) \mid x \in X\}$. $(S,d,r) \in \Sigma^n$ is said to be *symmetric* if every $\pi \in \Pi(N)$ satisfies $\pi(S) = S$, $\pi(d) = d$, and $\pi(r) = r$.

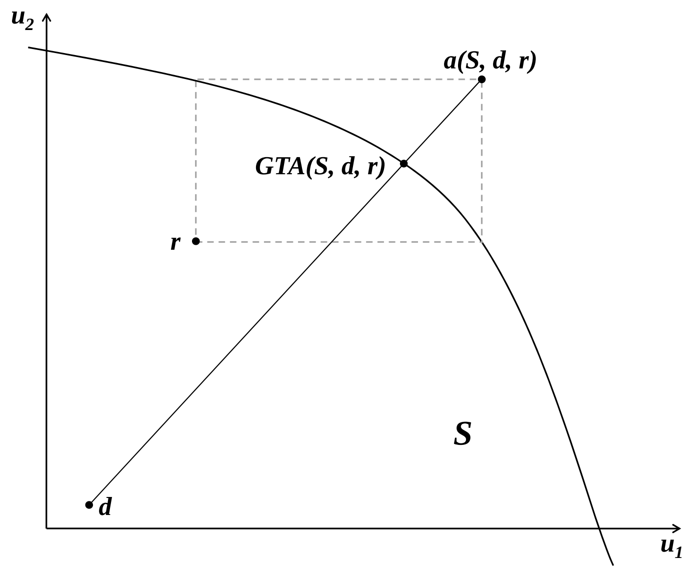

Figure 1: The GTA solution when $r \in S$

Symmetry (SYM): A solution ϕ on Σ^n satisfies SYM if for every symmetric problem (S, d, r) we have $\phi_i(S, d, r) = \phi_j(S, d, r)$ for all $i, j \in N$.

Let $\mathcal{L}$ be the family of vectors of functions $L = (L_i)_{i \in N}$ such that for every $i \in N$, $L_i : \mathbb{R} \longrightarrow \mathbb{R}$ and there exist $m_i \in \mathbb{R}_{++}$ and $b_i \in \mathbb{R}$ such that $L_i(t) = m_i t + b_i$ for every $t \in \mathbb{R}$.

Invariance under Positive Affine Transformations (IPAT): A solution ϕ on Σ^n satisfies IPLT if $\phi(L(S), L(d), L(r)) = L(\phi(S, d, r))$ for any $L \in \mathcal{L}$ and any $(S, d, r) \in \Sigma^n$.

The following axioms are analogous to well-known properties in the literature.

Restricted Monotonicity with respect to Aspirations (RMA): A solution ϕ on Σ^n satisfies RMA if, given any $(S,d,r),(S',d,r) \in \Sigma^n$, $S \subseteq S'$ and $a(S,d,r) = a(S',d,r)$ imply $\phi(S,d,r) \leq \phi(S',d,r)$.

Limited Sensitivity to Changes in Reference Point (LSCR): A solution F on Σ^n_r satisfies LSCR if for every $(S,d,r),(S',d',r') \in \Sigma^n_r$ if $S = S'$, $d = d'$, and $a(S,d,r) = a(S,d,r')$, then $F(S,d,r) = F(S',d',r')$.

Before going on, let us consider an example that shows the GTA solution is discontinuous. In what follows, limits are calculated relative to the Hausdorff topology.

Example 4.1 *The GTA solution is discontinuous. For example, for every integer* $k > 0$ *define the set* $S_k \subseteq \mathbb{R}^2$ *as* $S_k = cch\{(1, 1 - \frac{1}{k})\}$. *Define also* $d = (0,0)$, $r = (\frac{1}{2}, 1)$, *and* $S = cch\{(1,1)\}$ *so that the sequence* $\{S_k\}$ *converges to* S. *Then* $\lim_{k\to\infty} GTA(S_k,d,r) = (\frac{1}{2}, 1)$ *but* $GTA(S,d,r) = (1,1)$.

The discontinuity of $GTA(S,d,r)$ is more a pathological case than the norm. It occurs under very particular conditions. At any $(S,d,r) \in \Sigma^n$ such that S is *non-level*, i.e. $WPO(S) = PO(S)$, or such that $r \notin WPO(S) \setminus PO(S)$, $GTA(S,d,r)$ is continuous. Together, the last two axioms require a solution concept to be continuous everywhere on Σ^n, except at problems with the salient point exactly on the frontier of S.

Internal Continuity (IC): A solution ϕ on Σ^n satisfies IC if every sequence $\{(S_k, d_k, r_k)\}_{k=1,2,\ldots} \subset \Sigma^n$ such that $\lim_{k\to\infty}(S_k, d_k, r_k) = (S, d, r)$ and $r_k \in S_k$ for all k, satisfies $\lim_{k\to\infty} \phi(S_k, d_k, r_k) = \phi(S, d, r)$.

External Continuity (EC): A solution ϕ on Σ^n satisfies EC if every sequence $\{(S_k, d_k, r_k)\}_{k=1,2,\ldots} \subset \Sigma^n$ such that $\lim_{k\to\infty}(S_k, d_k, r_k) = (S, d, r)$, $r_k \notin S_k$ for all k, and $r \notin S$, satisfies $\lim_{k\to\infty} \phi(S_k, d_k, r_k) = \phi(S, d, r)$.

We now show that the axioms above uniquely characterize the GTA solution.

Proposition 4.2 *A solution ϕ on Σ^n satisfies WPO, SYM, IPAT, LSCR, RMA, EC and IC if and only if $\phi(S, d, r) = GTA(S, d, r)$ for any $(S, d, r) \in \Sigma^n$.*

Proof. It is simple to verify that GTA satisfies the axioms. Let ϕ be any solution that satisfies the axioms. If $r \notin S$, then RMA coincides with the axiom of *strong monotonicity*, so the characterization shown in (Chun and Thomson 1992) gives us the result. If $r \in S$, then we refer the reader to (Balakrishnan, Gómez, and Vohra 2011). ■

5 Connection to Other Solution Concepts

In this section we introduce other concepts in the literature. We divide our analysis according to whether the salient point is a feasible outcome of the negotiation or not.

5.1 Interior Salient Points

The case of a salient point inside the bargaining set has been analyzed with Gupta and Livne's (1988) *Bargaining Problem with a Reference Point.* They also proposed the solution concept defined as the maximal point (in S) that lies along the line joining the salient point to the ideal point. Here is the formal definition.

Definition 5.1 *For any problem $(S, d, r) \in \Sigma^n$ such that $r \in S$, the Gupta-Livne solution is defined as $GL(S, d, r) = \lambda^* z(S, d) + (1 - \lambda^*)r$ where $\lambda^* = \max\{\lambda \in [0, 1] \mid \lambda z(S, d) + (1 - \lambda)r \in S\}$.*

The GTA solution is "dual" to the Gupta-Livne solution in the sense that it exchanges the roles played by the salient and disagreement points. We use the salient point r to set bargainers' expectations (via aspirations) and the disagreement point d as a reference vector from which proportional payoffs are measured. Instead, they set expectations using d (via the ideal point) and measure proportional payoffs relative to r. The two solution concepts coincide if $r = d$ or if r is a Pareto optimal vector in S.

5.2 Exterior Salient Points

Chun and Thomson (1992) propose the main model and solution used to study the case when the salient point is outside the bargaining set. They call their model the *Bargaining Problem with Claims.* Although they require that $r \notin S$ and

$r \leq z(S, d)$, it is simple to extend their *Proportional solution* to our model. If the disagreement point is normalized to be at the origin, their concept assigns payoffs in proportion to the salient point.

Definition 5.2 *For any problem $(S, d, r) \in \Sigma^n$ such that $r > d$, define its Proportional solution as $P(S, d, r) = \lambda^* r + (1 - \lambda^*) d$ where $\lambda^* = \max\{\lambda \in \mathbb{R}_+ \mid \lambda z(S, d) + (1 - \lambda) r \in S\}$.*

We refer the reader to Section 8.2 in Thomson (2003) for a review of the related literature.

5.3 Arbitrary Salient Points

The *Rights Egalitarian solution* (Herrero, Maschler, and Villar 1999), a concept that evenly distributes monetary gains or losses, allows for both interior and exterior salient points. Herrero and Villar (2010) extends the concept, in two different ways, to a setting without transferable utility. Here is the natural adaptation of their single-valued solution to our environment.[1]

Given a problem $(S, d, r) \in \Sigma^n$ such that $r \leq z(S, d)$ and $r \notin S$, define the *minimum expectations point* as $m_i(S, d, r) = \max\{t \in \mathbb{R} \mid (t, r_{-i}) \in S\}$. The minimum expectations point can be interpreted as a "dual" to the aspirations point when $r \notin S$.

Definition 5.3 *The NTU Rights Egalitarian solution is defined for every $(S, d, r) \in$*

[1] Herrero and Villar (2010) actually call their concept the Proportional solution, but we will call it *NTU Rights Egalitarian solution* to prevent confusion with Chun and Thomson (1992).

Σ^n *such that* $d \leq r \leq z(S,d)$ *as*

$$RE(S,d,r) = \begin{cases} \lambda^* a(S,d,r) + (1-\lambda^*)r & \textit{if } r \in S \\ \mu^* r + (1-\mu^*) m(S,d,r) & \textit{if } r \notin S \end{cases}$$

where $\lambda^* = \max\{\lambda \in [0,1] \mid \lambda a(S,d,r) + (1-\lambda)r \in S\}$ *and* $\mu^* = \max\{\mu \in [0,1] \mid \mu r + (1-\mu)m(S,d,r) \in S\}$.

Herrero and Villar (2010) proposes a bargaining model called an *NTU Sharing Problem*. However, their model does not include a disagreement and a salient point but just an *entitlement* vector $c \in \mathbb{R}^n$. This means their solution only depends on the bargaining set S and just one other point. Not surprisingly, entitlements end up playing the role of the disagreement point. In fact, if $c \in S$, their solution concepts (and axiomatizations) coincide with those presented in Kalai and Smorodinsky (1975) and Nash (1950). We, on the other hand, focus on the interaction between the disagreement and salient points. Our model is more general because it is based on the assumption that the disagreement and salient points are essentially different.

The Proportional solution coincides with the GTA solution when $r \notin S$. Therefore, comparing other exterior-point concepts with our solution entails studying their relation to the Proportional solution.

5.4 Gains vs Losses

We now relate our solution concept to Kahneman and Tversky's (1979) "Prospect Theory." Applied to negotiations, their theory implies decision makers are risk

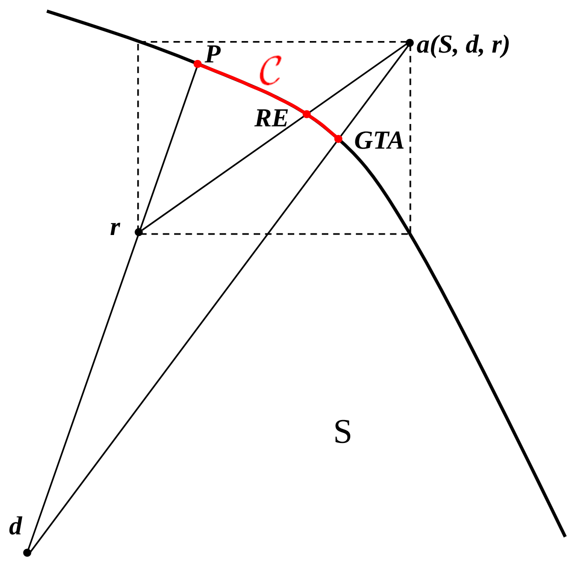

Figure 2: A cross section of S

averse, and consequently more likely to make concessions, when faced with the prospect of gains. However, when dealing with losses, bargainers are less willing to compromise (e.g. see Farber and Katz (1979), Neale, Huber, and Northcraft (1987) and Butler (2008)). If we interpret the salient point r as the status quo, the GTA solution handles gains ($r \in S$) differently from losses ($r \notin S$). Indeed, it divides losses proportionally to r, but divides gains more equally, in a more similar fashion to the NTU Rights Egalitarian solution.

Let $(S, d, r) \in \Sigma^n$ such that $r \in S$ and $r > d$ so that $P(S, d, r)$ is well defined. We will argue that $RE(S, d, r)$ is in a sense "closer" to $GTA(S, d, r)$ than $P(S, d, r)$ (see Figure 2). Assume for the sake of argument that $GTA(S, d, r) \notin \{RE(S, d, r), P(S, d, r)\}$, otherwise there is nothing to show. Then $r \neq a(S, d, r)$

and the points r, d and $a(S, d, r)$ are not collinear. Therefore $RE(S, d, r) \neq P(S, d, r)$. The set $\mathcal{C} = WPO(S) \cap con\{d, P(S, d, r), a(S, d, r)\}$ is homeomorphic to the closed interval $[0, 1]$ with extremes at $GTA(S, d, r)$ and $P(S, d, r)$. Now, $RE(S, d, r) \in \mathcal{C}$ but is different from its two extremes. Thus, moving along $\mathcal{C}$, $GTA(S, d, r)$ is closer to $RE(S, d, r)$ than to $P(S, d, r)$. This property of the GTA solution makes us optimistic about its predictive power when tested experimentally.

6 Concluding Remarks

Table 1 above lists in chronological order a number of contributions to the axiomatic bargaining literature that include a salient point akin to our r. Each work has been classified according to three parameters: 1) Ability to transfer utility among bargainers 2) Location of the salient point r with respect to the bargaining set S and 3) Existence and variability of the disagreement point d. Every paper in the list, except those omitting d, is a particular case of a Bargaining Problem with a Salient Point. It is interesting to notice that the only concept that the only solution concept on the the list that corresponds to the more general type of model under the three criteria, is the Generalized Tempered Aspirations solution.

The most important contribution of this work is to propose a more general setting that encompasses bargaining problems with claims and with reference points. Up to now, these two branches of the literature have evolved independently. Our model allows solution concepts in both fields to be compared, generalized using a common framework.

Article	TU / NTU	**r**	**d**
Nash (1950)	NTU	no r	variable d
O'Neill (1982)	TU	$r \notin S \setminus PO(S)$	$d = 0$
Moulin (1987)	TU	$r \in S$	$d = 0$
Young (1988)	TU	$r \notin S \setminus PO(S)$	$d = 0$
Chun (1988)	TU	$r \in \mathbb{R}^n_+$	$d = 0$
Gupta and Livne (1988)	NTU	$r \in S$	variable d
Chun and Thomson (1992)	NTU	$r \notin S$	variable d
Dagan and Volij (1993)	TU	$r \notin S \setminus PO(S)$	$d = 0$
Herrero (1998)	NTU	$r \notin S$	no d
Herrero, Maschler, and Villar (1999)	TU	$r \in \mathbb{R}^n$	no d
Herrero and Villar (2010)	NTU	$r \in \mathbb{R}^n$	no d
Balakrishnan, Gómez, and Vohra (2011)	NTU	$r \in \mathbb{R}^n$	variable d

Table 1: Classification of the axiomatic bargaining literature

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